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# An extension of Neumann's method for shape control of force-induced elastic vibrations by eigenstrains

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**Abstract**

This paper is concerned with the force-induced vibrations of linear elastic solids and structures. We seek a transient distribution of actuating stresses produced by additional eigenstrain, such that the vibrations produced by a given set of imposed forces are exactly compensated. This problem, known as dynamic shape control problem in structural engineering, or as dynamic displacement compensation problem in automatic control, is inverse to the usual direct problem of determining displacements due to imposed forces and actuation stresses. In the present paper, we extend a method, which was introduced by F.E. Neumann for demonstrating the uniqueness of direct elastodynamic problems. We use this extended Neumann method in order to show that the distribution of the actuating stresses for shape control must be equal to any statically admissible stress distribution that is in temporal equilibrium with the imposed forces. We furthermore discuss the role of stresses corresponding to this class of solutions in some detail, emphasizing the non-unique nature of a statically admissible stress. As an analytical justification of our formulations, we show that our method reveals some static results by J.M.C. Duhamel and by W. Voigt and D.E. Carlson. Particularly, our method can be interpreted as a dynamic extension of the Duhamel body-force analogy. We moreover present numerical results for a dynamically loaded, irregularly shaped domain in a state of plane strain. These finite element computations give excellent evidence for the validity of the presented method of shape control for both, the case of a step-input and the case of a harmonic excitation.


*Keywords:* Shape control; Force-induced vibrations; Eigenstrains; Thermal expansion strains; Piezoelectricity; Smart structures; Non-uniqueness

## 1. Introduction

Consider a solid structure that is fixed at some part of its boundary, and that is initially at rest in an undeformed state. The constitutive behavior of this body is assumed to be elastic and anisotropic. The body now is loaded by transient imposed forces, which are considered as known throughout the paper. Due to the imposed forces, vibrations of the body are induced. These force-induced vibrations may considerably

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lower the structural usability, particularly, when the body is driven into resonant vibrations. We therefore wish to superpose a transient distribution of actuation stresses, such that the force-induced vibrations are cancelled out exactly. In the following, we describe the vibrations of the body in the framework of the linear theory of elastodynamics, and we assume that the actuation stresses are produced by eigenstrains.

We present the following short review on eigenstrains. The name "eigenstrain" has been introduced by Mura (1991) in order to denote incompatible strains such as thermal expansion strains or plastic misfit strains. This name stems from the German word "Eigenspannungsquelle" (see Reißner, 1931). Contrary to imposed forces, eigenstrains under certain instances may be associated with deformation-free stresses, or with stress-free deformations. Thermoelasticity probably represents the most prominent example of eigenstrain type problems, see e.g. Parkus (1976) and Nowacki (1975) for some comprehensive expositions including dynamic formulations. There exists a well-known analogy between the actuating effect of thermal expansion in a thermoelastic body and the converse piezoelectric effect in a piezoelastic body, such that computational methods of thermoelasticity may be used to solve problems of piezoelectrically actuated bodies (see Vinson (1993)). Both, temperature and electric field are coupled to structural deformations, the latter effect being denoted as the direct piezoelectric effect and being utilized in practice for the sake of sensing. In the last decades, piezoelectricity has been extensively applied for actuation and sensing of structural vibrations (see Saravanos and Heyliger, 1999 and Rao and Sunar, 1999 for reviews). Frequently, the piezoelectric effects are realized in practice by means of piezoelastic stackers, patches or layers, which are bonded or otherwise integrated into the bodies to be actuated or sensed. It has become customary to summarize structures with integrated actuators and sensors, which are often connected by an automatic control system, under the notions of smart or intelligent materials and structronic systems (see Tzou, 1998 and Flatau and Chong, 2002 for reviews). Some of the various physical effects utilized in modern smart materials can be classified as eigenstrains. This is also true for the case of actuation by pre-stress, e.g. by active tendons.

As has been already mentioned, it is the scope of the present paper to cancel out exactly force-induced vibrations by means of actuation stresses produced by eigenstrains. In other words, our goal is to derive a transient distribution of eigenstrain-type actuating stresses, such that the vibrations produced by a known set of imposed forces are exactly compensated. Such a problem is inverse to the usual direct problem of determining the displacements due to given imposed forces and actuation stresses. In structural engineering, this inverse problem is known as a dynamic shape control problem, a term not to be confused with automatic control. An automatic or feedback control algorithm is needed when the time evolution of the imposed forces is not known in advance, when the vibrations would not start from an undeformed state of rest, or when there are other uncertainties. In the automatic control community, our topic would be denoted as compensation of the displacements induced by a known set of imposed forces.

In structural engineering, the field of shape control by eigenstrains started with a study by Haftka and Adelman (1985), who developed a procedure for determining the temperature in control elements so as to minimize the overall static distortion of a large space structure from its original shape. It has been noted by Haftka and Adelman (1985) that the disturbances, which affect the shape of structures, may be subdivided into two parts. One type is transient, while the second type of disturbances is due to fixed deformations or those, which are slowly varying in time. The present paper deals with transient type of disturbances, particularly with vibrations induced by imposed forces. A review on static and dynamic shape control by piezoelectric actuation has been presented by Irschik (2002), where eigenstrain type actuations other than the converse piezoelectric effect have been shortly addressed as well.

In the following, we treat the above stated shape control problem of compensating force-induced vibrations in the framework of the linear theory elastodynamics with eigenstrains (see e.g. Gurtin, 1972 and Carlson, 1972 for fundamental expositions). See also Irschik et al. (1993) for integral statements connecting force-induced and eigenstrain-induced vibrations. We assume that the eigenstrains or, equivalently, the actuation stresses produced by eigenstrains, can be applied everywhere within the body and throughout the whole time-period under consideration. Under these assumptions, a class of solutions of the shape control problem has been given by the present authors (2001), where use has been made of convolution-type integral theorems of elastodynamics, such as an extension of Graffi's theorem, see also our exposition (2003). The formulations given in Irschik and Pichler (2001, in press), extend a static solution presented by Irschik and Ziegler (2001). The work of our group on shape control of beam vibrations using integral statements has been recently extended by Irschik et al. (2003) to the case of electromechanically coupled beam vibrations. For preceding contributions of our group including cooperation to automatic control (see Irschik, 2002). For a use of the proposed shape control solution of force-induced beam vibrations in automatic control (see Schlacher and Kugi, 1999 and Kugi, 2001). Recently, shape control of plate vibrations has been treated by means of integral formulations by Nader et al. (2003).

The goal of the present contribution is, first, to present an alternative proof of results, which have been derived by Irschik and Pichler (2001) by means of integral theorems. In the present contribution, we extend a method originally used by Franz E. Neumann in order to prove the uniqueness of the solutions of direct problems of linear elastodynamics, i.e. of the problem of determining displacements and stresses in an elastic body due to given forces and, possibly, given actuation stresses. For a contemporary presentation of Neumann's method (see Chandrasekharaiah and Debnath, 1994). Our present problem of determining the actuation stresses such that force-induced vibrations are compensated is inverse to the cited direct problem, and a unique solution for shape control is therefore not to be expected in general. Extending Neumann's method, we are however able to show that, in order that our goal of shape control is achieved, the distribution of the actuating stresses must be equal to any statically admissible stress distribution that is in temporal equilibrium with the imposed forces. Our solution thus is not only easy to obtain, but it also explicitly reflects the non-uniqueness of the inverse problem under consideration. Our present derivations in principle lead to the same results as in Irschik and Pichler (2001). However, the extension of Neumann's method to the present context of shape control is felt to be more straightforward and to be a result of its own right. We moreover lay emphasis upon the role of stresses corresponding to the derived class of solutions of shape control, and we discuss, how the non-uniqueness of the proposed solutions may be used in order to decrease the amount of necessary actuation and thus the stresses. As an analytical justification of our formulations, we show that our method reveals some static results by J.M.C. Duhamel and by W. Voigt and D.E. Carlson. We finally present numerical results for a harmonically excited irregularly shaped plate. These results, produced by the finite element code Abaqus, give excellent evidence for the validity of the presented method of shape control in a wide frequency range.

## 2. Elastodynamic initial boundary value problem with actuation stresses

Consider a solid body $B$ under the action of imposed body forces $b$ per unit volume. The part $\partial B_1$ of the boundary of $B$ is assumed to be fixed in space, such that a rigid body motion of $B$ is prohibited. The remaining part of the boundary, $\partial B_2$, is loaded by imposed surface tractions $s$ per unit area. Both, the body forces and the surface tractions are assumed to be known. They may vary with time, such that the particles of $B$ in general will be accelerated, and time-dependent displacements and deformations will take place. In order to describe this transient deformation, we use an undeformed reference configuration, the placement of which is defined by the fixed part $\partial B_1$ of the boundary. We utilize the material description of continuum mechanics, in which all of the mechanical quantities are described as a function of the position vectors $p$ of the particles in this reference configuration and of time $t$. For the sake of simplicity, we use a common Cartesian co-ordinate system with unit vectors $e_i$, $i = 1, 2, 3$ when referring to the component form of vectors and tensors. In this common co-ordinate system, using Einstein's convention of summation about repeated indices, the position vector is written as $p = x_i e_i$.

The displacement vector of a particle, connecting its place in the reference configuration with its actual place, is described as

$$u = u(p,t) = u_i(p,t)e_i. \tag{1}$$

The gradient of $u$ with respect to the place $p$ in the reference configuration is a tensor of second order, which reads in the common co-ordinate system

$$\operatorname{grad} u = \frac{\partial u_i}{\partial x_j} e_i \otimes e_j. \tag{2}$$

The symbol $\otimes$ denotes the tensorial product of two vectors. The divergence of $u$ is

$$\operatorname{div} u = \operatorname{tr} \operatorname{grad} u = \frac{\partial u_i}{\partial x_i}. \tag{3}$$

The trace of a second order tensor is denoted by tr.

Throughout the paper we assume that the displacement $u$ and the tensor grad $u$ are small in the sense of the infinitesimal theory of continuum mechanics. Thus, we do not need to distinguish between the various measures of stress and strain defined with respect to the reference and the actual configuration. We furthermore assume linear relations between stress and strain to hold. Our formulations thus remain within the linear theory of elasticity. For foundations, we refer to Gurtin (1972), Carlson (1972) and Chandrasekharaiah and Debnath (1994).

The goal of the present contribution is to derive a distribution of actuation stresses produced by eigenstrains, which, when superposed upon the imposed body forces and surface tractions, results in zero total displacements $u$ throughout the body and at every time instant $t$. We note that the actuation stresses might be coupled to the structural deformation, e.g. by the direct piezoelectric effect, or by the rate of strain term in the heat conduction equation. This question however does not come into the play here, since we only ask for the necessary distribution of actuation stresses. We return to this point below, at the end of Section 6. We furthermore assume that the actuation stresses can be applied in a distributed manner everywhere within the body and throughout the whole time-period under consideration. In practice, spatially discretized actuators, e.g. discrete patches, have to be often used. There is an ongoing research of our group showing that a suitable placement of such discrete actuators can be found from the results derived for shape control assuming spatially distributed actuation. Moreover, spatially distributed, shaped actuators more and more are brought into practical applications (see the review given in Irschik, 2002).

As a first step, we state the initial boundary-value problem in the framework of which the goal of compensating force induced displacements is to be achieved. We start with Cauchy's first law of motion, which we write as

$$B: \rho \ddot{u} = b + \operatorname{div} S^{\mathrm{T}}. \tag{4}$$

The density of mass is denoted by $\rho$, and a superimposed dot stands for the material time derivative, $\dot{u} = \partial u(p,t)/\partial t$. Furthermore, $S = S_{ij} e_i \otimes e_j$ is the stress tensor, and the superscript T denotes the transpose of a tensor. We assume imposed couples as well as couple stresses to be absent, such that the stress tensor becomes symmetric: $S^{\mathrm{T}} = S$. In Eq. (4), the divergence term then can be written as

$$\operatorname{div} S^{\mathrm{T}} = \operatorname{div} S = \frac{\partial S_{ij}}{\partial x_j} e_i. \tag{5}$$

The stress is taken to be related to the strain by a generalization of Hooke's law in the form

$$B: S = C[E] + S^a, \tag{6}$$

where the strain $E$ in the infinitesimal theory is related to $\operatorname{grad} u$ of Eq. (2) by

$$B: E = \operatorname{sym} \operatorname{grad} u, \tag{7}$$

the symmetric part of a tensor being abbreviated by sym. The fourth-order tensor of elastic moduli is denoted by $C$, and $C[E]$ stands for the second-order tensor that represents the linear mapping of $E$ by means of $C$:

$$C[E] = C_{ijkl}E_{kl}e_i \otimes e_j. \tag{8}$$

In the case of an isotropic material Eq. (8) is given by

$$C[E] = 2\mu E + \lambda I \operatorname{tr} E, \tag{9}$$

where $I = e_i \otimes e_i$ is the identity tensor, and $\mu$ and $\lambda$ are the two Lamé moduli. Furthermore, the tensor of second order $S^a = S^{a^{\mathrm{T}}}$ in Eq. (6) denotes an actuation stress, being the result of eigenstrain acting at the particle under consideration, e.g., in case of a thermal loading of a thermoelastic body with isotropic linear thermal expansion, there is

$$B: S^a = C[\alpha]\theta, \tag{10}$$

where $\theta$ is the increase in temperature, and $\alpha$ denotes the second-order tensor of coefficients of linear thermal expansion. In the isotropic case, there is $C[\alpha] = -(3\lambda + 2\mu)\hat{\alpha}I$; the thermal eigenstrain $\hat{\alpha}\theta$ is also denoted as thermal expansion strain in the literature.

In the sequel of our derivations, we consider anisotropic material behavior. The following relation of symmetry nevertheless must hold for $C$:

$$C[E] \cdot \overline{E} = C[\overline{E}] \cdot E, \tag{11}$$

where $\overline{E} = \overline{E}^{\mathrm{T}}$ denotes a symmetric tensor of second order, and the dot product indicates the double contraction of two second-order tensors to a scalar quantity. As an example for the dot product of two tensors of second order, consider the stress power given by

$$S \cdot \dot{E} = \operatorname{tr}(S^{\mathrm{T}}\dot{E}) = S_{ij}\dot{E}_{ij}. \tag{12}$$

The set of field equations governing our problem is formed by Eqs. (4), (6) and (7), where we assume a sufficient continuity of the fields under consideration to be guaranteed. Additionally, at the part $\partial B_1$ of the boundary of $B$, we have the boundary condition of place:

$$\partial B_1: \ u = 0, \tag{13}$$

while at $\partial B_2$ there is the boundary condition of traction

$$\partial B_2: \ S^{\mathrm{T}}n = Sn = s. \tag{14}$$

The vector $Sn$ represents the linear transformation of the vector $n$ by means of the second-order tensor $S$:

$$Sn = S_{ij}n_j e_i. \tag{15}$$

The boundary condition stated in Eq. (14) follows from Cauchy's fundamental theorem on stresses, when written for a particle at the boundary $\partial B_2$ with the outward unit normal vector $n$.

Since we assume that the vibrations start from an undeformed state of rest, we furthermore consider homogeneous initial conditions, such that the initial displacements and velocities vanish everywhere in $B$,

$$t = 0: \ u = 0, \ \dot{u} = 0. \tag{16}$$

Using Eq. (7), it is seen that this latter relation is associated with vanishing initial values of strain and rate of strain,

$$t = 0: \ E = 0, \ \dot{E} = 0. \tag{17}$$

In 1885, Franz E. Neumann proved the uniqueness of the above elastodynamic problem for the case of an isotropic material with $S^a = 0$. He first assumed two solutions to be produced by the same set of body forces and surface tractions. The difference of these two solutions then must be represented by an elastodynamic problem with zero body forces and surface tractions. In extension of a previous static consideration by Kirchhoff, Neumann showed that the latter problem can have only a trivial (vanishing) solution. For a contemporary presentation of Neumann's proof, see Chandrasekharaiah and Debnath (1994). In the next section, we assume $b$ and $s$ to be known in advance, and we use the technique developed by Neumann in order to derive additional distributions of $S^a$ such that there result zero total displacements $u$ throughout the body and at every time instant. As was mentioned in Section 1, this shape control problem in general does not have a unique solution.

## 3. Exact compensation of force-induced displacements by actuation stresses

Motivated by the cited derivation of Neumann, we define the following function of time

$$N(t) = \int_B (C[E] \cdot E + \rho \dot{u} \cdot \dot{u}) \, \mathrm{d}V \tag{18}$$

by integration over the body $B$. The scalar vector product

$$\dot{u} \cdot \dot{u} = \sum_{i=1}^{3} \dot{u}_i^2 \tag{19}$$

gives the square of the velocity vector of a particle.

Differentiating Eq. (18) with respect to time yields

$$\dot{N} = 2 \int_B (C[E] \cdot \dot{E} + \rho \dot{u} \cdot \ddot{u}) \, \mathrm{d}V, \tag{20}$$

which follows from the symmetry condition for the tensor of elastic moduli (Eq. (11)). From the generalized Hooke's law (Eq. (6)) from the kinematic relation given in Eq. (7), and since $S$ and $S^a$ are symmetric, we obtain that

$$C[E] \cdot \dot{E} = (S - S^a) \cdot \dot{E} = (S - S^a) \cdot \operatorname{sym} \operatorname{grad} \dot{u} = (S - S^a) \cdot \operatorname{grad} \dot{u}. \tag{21}$$

The following tensorial identity holds:

$$(S - S^a) \cdot \operatorname{grad} \dot{u} = -\operatorname{div}(S - S^a) \cdot \dot{u} + \operatorname{div}((S - S^a)\dot{u}). \tag{22}$$

Inserting Eqs. (22) and (21) into Eq. (20) and using the divergence theorem, it is found that

$$\dot{N} = -2 \int_B \operatorname{div}(S - S^a) \cdot \dot{u} \, \mathrm{d}V + 2 \int_{\partial B} (s - S^a n) \cdot \dot{u} \, \mathrm{d}S + 2 \int_B \rho \dot{u} \cdot \ddot{u} \, \mathrm{d}V. \tag{23}$$

The following identity is easily verified:

$$(S - S^a)\dot{u} \cdot n = (S - S^a)n \cdot \dot{u}. \tag{24}$$

Hence, due to Cauchy's fundamental theorem on stresses (Eq. (14)) and from Cauchy's first law of motion (Eq. (4)) we can write Eq. (20) as

$$\dot{N} = 2 \int_B (b + \operatorname{div} S^a) \cdot \dot{u} \, \mathrm{d}V + 2 \int_{\partial B} (s - S^a n) \cdot \dot{u} \, \mathrm{d}S. \tag{25}$$

In order to proceed further with Eq. (25), we define a statically admissible stress field $\widehat{S}$, which satisfies the static field equation

$$B : \operatorname{div} \widehat{S} = -b, \tag{26}$$

as well as the following boundary condition at $\partial B_2$:

$$\partial B_2 : \widehat{S} n = s. \tag{27}$$

Note that Eq. (26) represents the quasi-static version of Cauchy's first law (Eq. (4)) i.e. it is an equilibrium condition. Note, furthermore, that Eq. (27) coincides with the boundary condition of traction (Eq. (14)). The quasi-static boundary value problem indicated in Eqs. (26) thus may have an infinite number of solutions, since we do not specify a constitutive relation, nor do we specify a boundary condition at $\partial B_1$.

Now suppose that $S^a$ coincides with a statically admissible stress field satisfying Eqs. (26) and (27),

$$S^a = \widehat{S}, \tag{28}$$

throughout $B$ and at every time instant under consideration. Then, inserting Eqs. (26) and (27) into Eq. (25), it is found that

$$\dot{N} = 0, \tag{29}$$

such that the $N$ must be constant, $N(t) = \text{const}$. Hence, inserting the initial conditions given in Eqs. (16) and (17) into Eq. (18), it is found that $N$ must vanish during the whole course of the motion:

$$N(t) = 0. \tag{30}$$

We now assume, as it is customary in the linear theory of elasticity, that for a non-vanishing strain, $E \neq 0$, there is

$$C[E] \cdot E > 0. \tag{31}$$

The tensor of elastic moduli thus is assumed to be strongly elliptic. Moreover, the mass density is positive, $\rho > 0$. It is thus seen that, in order to satisfy Eq. (30), the integrand in Eq. (18) must be zero. Consequently, the strain $E$ and the velocity $\dot{u}$ must vanish throughout $B$ and at every time instant in order that Eq. (30) can be satisfied. We have thus arrived at the following theorem:

Assume that the tensor of elastic constants is strongly elliptic (Eq. (31)) and that the initial conditions are homogeneous (Eq. (16)). Then, given an actuation stress satisfying Eq. (28), the force-induced dynamic displacements will be compensated, such that

$$B : u(p, t) = 0. \tag{32}$$

Note that $\widehat{S}$ in Eq. (28) follows from the solution of a quasi-static boundary value problem, which is much simpler to handle than the original initial-boundary-value problem stated in Section 2, and which reveals the non-uniqueness of the shape control problem under consideration. This non-uniqueness by no means represents a drawback in the present context. Contrary, this fact may be used to decrease the necessary effort for actuation, as is shortly shown below.

## 4. Stresses associated with an exact compensation of force-induced displacements

Compensation of force-induced vibrations of course does not mean to compensate the corresponding force-induced stresses. However, the total stresses due to the application of both, forces and eigenstrain, show an interesting behavior, and they may be influenced in a beneficial manner due to the non-uniqueness of the underlying solution of shape control presented above. First, note that every divergence-free stress field with zero surface tractions on $\partial B_2$ may be superimposed upon Eq. (28) without leading to non-zero displacements. Indeed, any stress distribution

$$S_+^a = \widehat{S} + \widehat{S}_0, \tag{33}$$

where $\widehat{S}$ is a solution of Eqs. (26) and (27), and $\widehat{S}_0$ satisfies the homogeneous set of equations

$$B: \ \operatorname{div} \widehat{S}_0 = 0, \tag{34}$$

$$\partial B_2: \ \widehat{S}_0 n = 0, \tag{35}$$

does represent another solution of our shape-control problem. But since the total displacements now vanish (Eq. (32)) this is also true for the strains (Eq. (7))

$$E(p,t) = 0. \tag{36}$$

Inserting Eqs. (33) and (36) into the constitutive equations (Eq. (6)) the following result thus is obtained for the stresses:

$$S = S_+^a = \widehat{S} + \widehat{S}_0. \tag{37}$$

Hence, and this is an interesting result we would like to emphasize, it is found that the stresses corresponding to the above solution of dynamic shape control are equal to the actuation stresses, which in turn are formed by the stresses corresponding to quasi-static problems. That is, to minimize the actuation stresses means to minimize the total stresses. It is important to note that, since the quasi-static problems stated in Eqs. (26) and (27) and Eqs. (34) and (35), respectively, are not complete, the stresses can be influenced accordingly, with the rare exception of statically determinate problems. In the latter, $\widehat{S}$ follows uniquely from Eqs. (26) and (27), such that there is $\widehat{S}_0 = 0$ in Eqs. (34) and (35). This case e.g. occurs when transferring the above solution of shape control from the three-dimensional theory to a statically determinated beam or truss by analogy. In general, however, a rational regularization strategy for obtaining a stress distribution $\widehat{S}$ satisfying Eqs. (26) and (27) is to add proper constitutive relations and boundary conditions of place on $\partial B_1$, and to compute the corresponding quasi-static stresses afterwards. These constitutive relations and boundary conditions of course need not to coincide with the original ones, see Eqs. (6) and (13), respectively. The same strategy can be followed for the divergence-less stress $\widehat{S}_0$ in Eqs. (34) and (35). Of course, the constitutive relations and boundary conditions for $\widehat{S}$ and $\widehat{S}_0$ need not to be the same. Moreover, the constitutive relations need not even to be elastic, and the boundary conditions of place need not to be homogeneous. This offers a wide range of tailoring $\widehat{S}$ and $\widehat{S}_0$ so as to minimize the stress $S$, and thus the actuation stress $S_+^a$, in Eq. (37). Such a strategy is planned to be studied in some detail in a future investigation. Note furthermore that the part of actuation formed by $S_+^a - S^a = \widehat{S}_0$ obviously does not produce any strain, since it can be added to $S^a$ in Eqs. (28) without violating the goal of shape control (see Eq. (37)). Such a solution has been termed a nil-potent solution in a previous static study by Irschik and Ziegler (2001).

In passing to the following numerical example, we now assume that $\widehat{S}$ and $\widehat{S}_0$ belong to two quasi-static boundary value problems that obey the original homogeneous boundary condition of place (see Eq. (13)). Furthermore, the corresponding constitutive relations are chosen to have the original linear form with the original tensor of elastic parameters $C$ (compare Eq. (6)). We then assume that $\widehat{S}$ belongs to a problem without actuation stresses,

$$B: \widehat{S} = C[\widehat{E}], \tag{38}$$

while $\widehat{S}_0$ is due to some actuation stresses $S_0^a$,

$$B: \widehat{S}_0 = C[\widehat{E}_0] + S_0^a, \tag{39}$$

with the corresponding quasi-static strains $\widehat{E}$ and $\widehat{E}_0$, respectively. When setting $S_0^a = 0$, there follows $\widehat{S}_0 = 0$, since $\widehat{S}_0$ is divergence-free (Eq. (34)) and there are no other sources of strain, such that one obtains $\widehat{E}_0 = 0$ in Eq. (39). Having computed $\widehat{S}$, we insert it into Eqs. (28) or (37), in order to form the actuation stress $S^a$. This choice has been used in the subsequent numerical example.

## 5. Some analytical verification

Before turning to numerical computations, however, we give some comparisons to the literature for the sake of an analytical verification of the above formulations. We start with two interesting cases stemming from the previous study on static shape-control by Irschik and Ziegler (2001). First, note that $\widehat{S}_0 = 0$ in Eq. (39) also follows from the non-trivial choice $S_0^a = \widehat{S}$, since then, as the result of applying the above dynamic considerations in the static limit, and as has been shown before by Irschik and Ziegler (2001), there is $\widehat{E}_0 = -\widehat{E}$, such that, $C[\widehat{E}_0] = -\widehat{S}$ (see Eqs. (38) and (39)). Such a solution that produces zero static stresses is called an impotent eigenstrain field in the literature (see Mura, 1991). On the other hand side, consider any eigenstrain loading $\overline{S}_0^a$ with a non-vanishing quasi-static stress $\widehat{\overline{S}}_0$, and then take $S_0^a = \widehat{\overline{S}}_0$ in Eq. (39). It can be shown that the static strains then vanish in Eq. (39), $\widehat{E}_0 = 0$ and $\widehat{S}_0 = \widehat{\overline{S}}_0$ (see Irschik and Ziegler, 2001). Such a solution, which produces zero static strains, is called a nil-potent solution, see also the remark given above, and it can be used to influence the total stresses due to both, forces and eigenstrain actuation.

For the sake of a further analytical justification, we consider the case of an eigenstrain stemming from a temperature rise $\theta$. The actuation stress $S^a$ then is given by Eq. (10). When $S^a$ is equal to a quasi-static stress $\widehat{S}$ that is in temporal equilibrium with a set of body forces $b$ and surface traction $s$ (Eqs. (26) and (27)) the displacements produced by $b$ and surface traction $s$ should be completely compensated by $\theta$ (see Eq. (28)). Conversely now, assuming $\theta$ to be given, the set of body forces

$$b = -\mathrm{div}\, S^a = -\mathrm{div}(C[\alpha]\theta) \tag{40}$$

and surface traction

$$s = S^a n = C[\alpha] n \theta \tag{41}$$

should compensate the displacements due to $\theta$. But, due to the linearity of the theory under consideration, the negative of the body forces and surface traction given in Eqs. (40) and (41) then should produce displacements that are equal to the displacements produced by the temperature rise $\theta$. In the static case, this result indeed is known as the body-force analogy of linear thermoelasticity, which dates back to J.C.M. Duhamel (see Section 11 of Carlson, 1972). Hence, it is seen that our formulation is in agreement with the static Duhamel body-force analogy. The derivations given in the present paper furthermore indicate that the Duhamel body-force analogy may be utilized also in the dynamic case. This will be elaborated in more detail elsewhere.

As a further analytical justification, we consider the case of a homogeneous body that is fixed everywhere at the boundary (see Eq. (13)) such that boundary conditions of traction are absent (see Eq. (14)). The body is assumed to be loaded by a constant static body force $b$. We again consider the case of an eigenstrain stemming from a temperature rise $\theta$, the actuation stress $S^a$ being given by Eq. (10). When this $S^a$ is equal to a quasi-static stress $\widehat{S}$ that is in equilibrium with $b$ (Eq. (26)) the displacements produced by $b$ should be completely compensated by $\theta$ (see Eq. (28). This leads to the requirement

$$\mathrm{div}\, S^a = \mathrm{div}(C[\alpha]\theta) = \mathrm{div}\, \widehat{S} = -b = \text{const.} \tag{42}$$

Since the body is homogeneous, $C[\alpha]$ is constant. This implies

$$C[\alpha]\, \mathrm{grad}\, \theta = -b = \text{const.} \tag{43}$$

Taking $C[\alpha]$ to be invertible, Eq. (43) can be integrated so as to obtain

$$\theta = -((C[\alpha])^{-1}b)\cdot p + \theta_0, \tag{44}$$

where $p$ is the position vector introduced in Section 2, and $\theta_0$ denotes an arbitrary temperature distribution. Indeed, this result dates back to W. Voigt and has been substantiated in Section 15 of Carlson (1972). Hence, our method is in coincidence with the static Voigt–Carlson result about a temperature field that induces a displacement-free state.

## 6. Numerical example

For a validation of the dynamic solution strategy for shape control discussed in Sections 3 and 4, the following example problem is considered. A polygonal domain (see Fig. 1) is in a state of plane strain, the non-dimensionalized co-ordinates of its corners are $P_1:(0.0/1.0)$, $P_2:(0.0/0.0)$, $P_3:(1.0/0.7)$ and $P_4:(1.2/0.2)$. The domain is fixed at the left boundary, $x=0$, the edges $P_2-P_4$ and $P_3-P_4$ are free of stress. At the edge $P_1-P_3$ the domain is loaded by a distributed surface traction in the form of a time dependent pressure, $s=-p=-\hat{p}q(t)$, with the constant pressure $\hat{p}$ and a given time-evolution $q(t)$.

In this example problem the required actuation for compensating the vibrations due to the external surface traction is produced by means of a specific distribution of thermal expansion strain according to Eq. (28). The coupling of thermal and mechanical fields is neglected. First, the quasi-static stress distribution due to the surface traction is computed as

$$\widehat{S} = \widehat{S}_{\text{stat}}q(t), \tag{45}$$

where $\widehat{S}_{\text{stat}} = C[\widehat{E}_{\text{stat}}]$ is the time-invariant static stress due to the constant surface pressure $\hat{p}$, the corresponding static strain being $\widehat{E}_{\text{stat}}$. The eigenstrain actuation then is computed in the form

$$S^a = \Lambda\hat{\theta}q(t) = C[\alpha]\hat{\theta}q(t) \tag{46}$$

with a constant reference temperature $\hat{\theta}$. The anisotropic tensor of stress-temperature coefficients $\Lambda$ in Eq. (46) is computed according to Eqs. (28) and (45) as

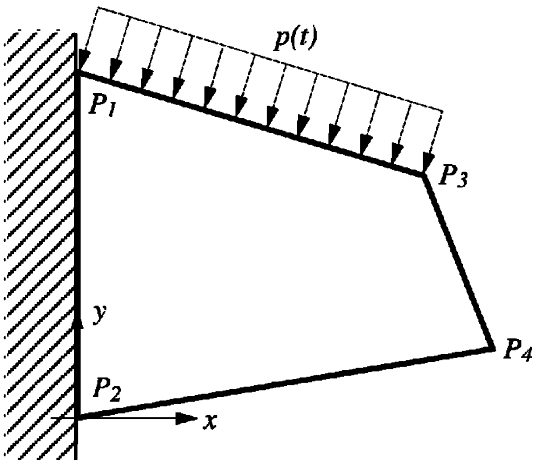


Fig. 1. Polygonal domain with a transient surface traction.

$$\Lambda = \widehat{S}_{\mathrm{stat}}\hat{\theta}^{-1}. \tag{47}$$

Equivalently, the tensor of thermal expansion coefficients $\alpha$ in Eq. (46) is taken in the anisotropic form

$$\alpha = \widehat{E}_{\mathrm{stat}}\hat{\theta}^{-1}. \tag{48}$$

The finite element validation was done using ABAQUS Standard 6.2 and a self-developed Visual C++ code for pre-processing. The polygonal domain was subdivided into 525 plane strain elements of type CPE4R for the computations, the material properties of steel were assigned. In a first step the static stress $\widehat{S}_{\mathrm{stat}} = C[\widehat{E}_{\mathrm{stat}}]$ was computed. Using the C++ code, the proper thermal expansion tensors $\alpha$ were calculated according to Eq. (48) and assigned to the corresponding finite elements. In the next step the finite element code was used to perform dynamic computations with the latter anisotropic thermal expansion tensors.

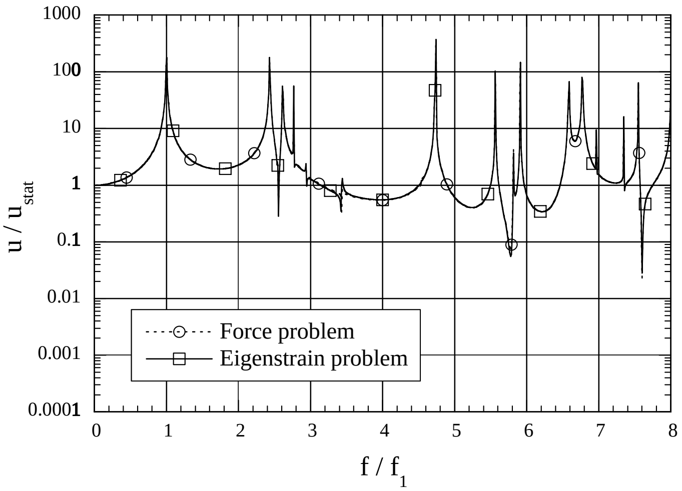


Fig. 2. Comparison of amplitude lags, horizontal displacement.

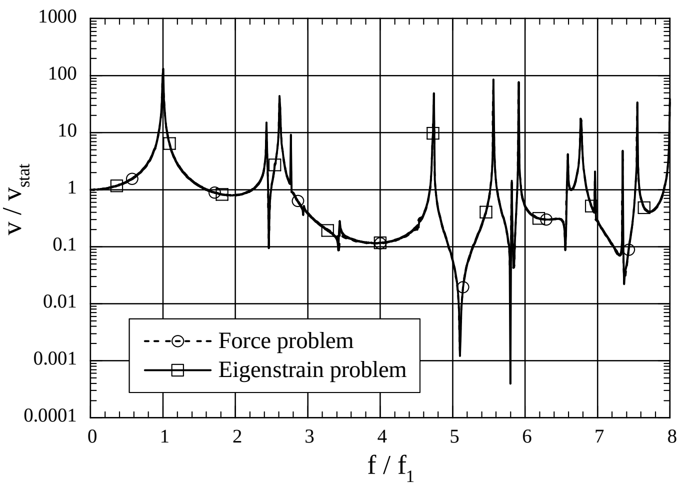


Fig. 3. Comparison of amplitude lags, vertical displacement.

Two loading cases were considered. The case of vibrations due to the surface tractions is called the force problem, and the case due to the negative actuation stresses given in Eqs. (46)–(48) was called the eigenstrain problem. Since we use the negative actuation stress, displacements in the force and the eigenstrain problem should be equal. Both, transfer functions in case of harmonic excitation, as well as step response functions were computed for the force problem and the (thermal) eigenstrain problem. As a characteristic result, the amplitude response spectra of the horizontal displacement $u$ and the vertical displacement $v$ of point $P_4$ in Fig. 1 are presented in Figs. 2 and 3, respectively. The displacements are scaled by means of the quasi-static displacements, and the excitation frequency $f$ is scaled by the fundamental eigenfrequency $f_1$ of the domain. The step response functions of the scaled horizontal and vertical displacement of point $P_4$ in the case of a suddenly applied load are presented in Figs. 4 and 5. The time in these figures is scaled by the fundamental vibration period $T_1$. Both the response spectra and the step responses of the force problem and

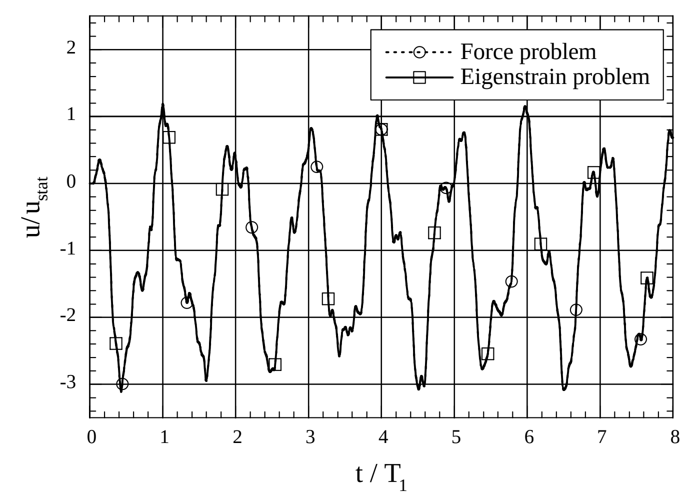


Fig. 4. Comparison of step responses, horizontal displacement.

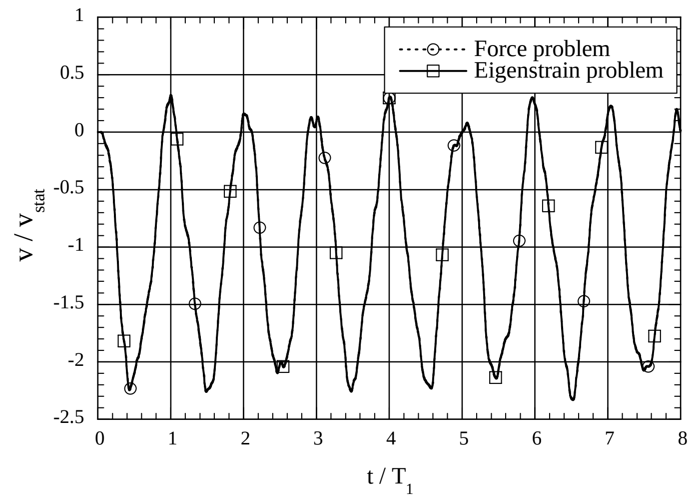


Fig. 5. Comparison of step responses, vertical displacement.

the thermal eigenstrain problem do coincide with a high accuracy and in a wide frequency range and time period, respectively, which gives excellent evidence for the validity of the presented class of solutions of the dynamic shape control problem.

## 7. Conclusion

In the present paper, we have presented a solution strategy for finding a distribution of eigenstrain-induced actuation stress such that the vibrations produced by given external forces do vanish. Extending a method dating back to F. Neumann, we have shown that any statically admissible stress that is in temporal equilibrium with the imposed forces, when it is applied as actuation stress in addition to the given forces, induces zero displacements throughout the body. This practically appealing solution strategy, namely that the engineer only has to solve a (quasi-) static problem in order to obtain an actuation stress for suppressing force-induced vibrations, should assure the significance of technical content of the paper. Our result has been confirmed by demonstrating a coincidence with analytical results from the literature, as well as by a numerical study. Of course, there remains the question, how to realize such an eigenstrain-induced actuation stress in practice. This seems to be a relatively easy task in the case of flexural vibrations of beams and plates, where the statically admissible stress can be computed analytically and applied practically in the form of shaped actuating layers, e.g. made of piezoelastic material. Our group has contributed rather intensively to this field, both analytically and experimentally. In dynamic two- and three-dimensional problems, however, numerical methods must be used (except in rare cases), and practical realizations remain an open area. The newly emerging fields of smart and functionally graded materials, which seek to tailor the material according to a desired purpose, nevertheless give much hope for practical applications in the very near future.

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